

# ITERATIONS OF THE MAP $X \mapsto \frac{1}{2}(X + X^{-1})$ OVER FINITE FIELDS OF ODD CHARACTERISTIC AND SEQUENCES OF IRREDUCIBLE POLYNOMIALS

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**ABSTRACT.** After studying the graphs associated with the map  $\vartheta(x) = \frac{1}{2} \cdot (x + x^{-1})$  over finite fields of odd characteristic, we construct infinite sequences of monic irreducible polynomials with coefficients in prime fields. We make no assumptions on the coefficients of the first polynomial  $f_0$  of the sequence, which belongs to  $\mathbf{F}_p[x]$ , for some odd prime  $p$ , and has positive degree  $n$ . If  $p^{2n} - 1 = 2^{e_1} \cdot m$  for some odd integer  $m$  and non-negative integer  $e_1$ , then, after an initial segment  $f_0, \dots, f_s$  with  $s \leq e_1$ , the degree of the polynomial  $f_{i+1}$  is twice the degree of  $f_i$  for any  $i \geq s$ .

## 1. INTRODUCTION

In [Ugo12] the iterations of the map  $x \mapsto x + x^{-1}$  over finite fields of characteristic 2 were studied, constructing, for a generic field  $\mathbf{F}_{2^n}$  with  $2^n$  elements, a graph whose vertices are labelled by the elements of  $\mathbf{F}_{2^n} \cup \{\infty\}$  and connecting a vertex  $\alpha$  with a vertex  $\beta$  if  $\beta = \alpha + \alpha^{-1}$ . Such a graph presents notable symmetries and a detailed description is given in [Ugo12]. Our study has been extended to finite fields of characteristic 3 ([Ugo11b]) and 5 ([Ugo11a]), where symmetries in the graphs are present too. The experimental evidence seems to suggest that the graphs associated with the map  $x \mapsto x + x^{-1}$  in finite fields of odd characteristic  $p > 5$  are trickier to study. Indeed, the resulting graphs seem not to present notable symmetries. Notwithstanding, if we slightly modify the map, the scenario is much more clear and a thorough description of the graphs is possible.

Let  $\mathbf{F}_q$  be a finite field with  $q$  elements, where  $q = p^n$  for some odd prime  $p$  and positive integer  $n$ . We can define a map  $\vartheta$  on  $\mathbf{P}^1(\mathbf{F}_q) = \mathbf{F}_q \cup \{\infty\}$  in such a way:

$$\vartheta(x) = \begin{cases} \frac{1}{2}(x + x^{-1}) & \text{if } x \neq 0, \infty \\ \infty & \text{if } x = 0 \text{ or } \infty \end{cases}$$

We introduce the following notation.

**Definition 1.1.** If  $m = 2^e \cdot k$ , for some odd integer  $k$  and non-negative integer  $e$ , then we denote by  $\nu_2(m)$  the exponent of the greatest power of 2 which divides  $m$ , namely  $\nu_2(m) = e$ .

The iterations of the map  $\vartheta$  over  $\mathbf{P}^1(\mathbf{F}_q)$  can be studied relying upon the consideration that  $\vartheta$  is conjugated to the square map. Indeed, if  $x$  is any element of  $\mathbf{P}^1(\mathbf{F}_q)$ , then

$$(1.1) \quad \vartheta(x) = \psi \circ s \circ \psi(x),$$

where  $s$  and  $\psi$  are maps defined on  $\mathbf{P}^1(\mathbf{F}_q)$  as follows:

$$s(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{F}_q \\ \infty & \text{if } x = \infty \end{cases} \quad \psi(x) = \begin{cases} \frac{x+1}{x-1} & \text{if } x \in \mathbf{P}^1(\mathbf{F}_q) \setminus \{1, \infty\} \\ 1 & \text{if } x = \infty \\ \infty & \text{if } x = 1 \end{cases}$$

Since  $\psi$  is a self-inverse map over  $\mathbf{P}^1(\mathbf{F}_q)$ , namely  $\psi^2(x) = x$  for any  $x \in \mathbf{P}^1(\mathbf{F}_q)$ , the following holds for the  $k$ -th iterate of  $\vartheta$ :

$$\vartheta^k(x) = \psi \circ s^k \circ \psi(x).$$

We say that an element  $x \in \mathbf{P}^1(\mathbf{F}_q)$  is  $\vartheta$ -periodic (resp.  $s$ -periodic) iff  $\vartheta^k(x) = x$  (resp.  $s^k(x) = x$ ), for some positive integer  $k$ . The smallest such  $k$  will be called the period of  $x$  with respect to the map  $\vartheta$  (resp.  $s$ ).

We can associate a graph  $Gr_q$  with the map  $\vartheta$  over  $\mathbf{P}^1(\mathbf{F}_q)$ . To do that, we label the vertices of the graph by the elements of  $\mathbf{P}^1(\mathbf{F}_q)$ . Then, if  $\alpha, \beta \in \mathbf{P}^1(\mathbf{F}_q)$  and  $\beta = \vartheta(\alpha)$ , we connect with an arrow  $\alpha$  to  $\beta$ .

In the following Section we will describe thoroughly the structure of the generic graph  $Gr_q$ . Such a graph is made up by a number of connected components, which is given in Theorem 2.3. Any connected component is formed by a cycle and any vertex of the cycle is the root of a reversed binary tree. The possible lengths of such cycles are given in the same Theorem, while the depth of the trees is given in Theorem 2.5.

In the subsequent Section we introduce a transformation  $\tilde{Q}$ , which takes a polynomial  $f \in \mathbf{F}_p[x]$  of degree  $n$  to

$$f^{\tilde{Q}}(x) = 2^n \cdot x^n \cdot f(\vartheta(x)).$$

Relying upon the  $\tilde{Q}$ -transform and the structure of the graph  $Gr_{p^n}$  we describe a possible procedure to construct infinite sequences of irreducible polynomials with coefficients in  $\mathbf{F}_p$  (see Theorem 3.9). Such sequences are constructed inductively. We take a monic irreducible polynomial  $f_0 \in \mathbf{F}_p[x]$  of degree  $n$ , making no assumptions on its coefficients. For  $i \geq 0$ , if  $f_i^{\tilde{Q}}$  is irreducible, then we set  $f_{i+1} := f_i^{\tilde{Q}}$ . Otherwise,  $f_i^{\tilde{Q}}$  factors as the product of two monic irreducible polynomials of the same degree and we set  $f_{i+1}$  equal to one of these two factors. If  $\nu_2(p^{2^n} - 1) = e_1$ , then, after an initial segment  $f_0, \dots, f_s$  of the sequence with  $s \leq e_1$ , the degree of  $f_{i+1}$  is twice the degree of  $f_i$  for  $i \geq s$ .

The procedure just described involves at most  $e_1$  polynomial factorizations. In Section 4 we describe a procedure to factor the generic polynomial  $f_i^{\tilde{Q}}$  into the product of two polynomials of the same degree. Such a factorization involves some reductions modulo  $f_i$  in  $\mathbf{F}_p[x]$ , the solution of a linear system of at most  $n$  equations with  $n$  unknowns over  $\mathbf{F}_p$  and the computation of a square root of an element of  $\mathbf{F}_p$ .

## 2. STRUCTURE OF THE GRAPHS

We fix once for all the current Section an odd prime  $p$  and a positive integer  $n$  and set  $q = p^n$ . Let  $\nu_2(q - 1) = e$  for some positive integer  $e$ .

**Lemma 2.1.** *Let  $\alpha \in \mathbf{P}^1(\mathbf{F}_q)$ . Then,  $\alpha$  is  $\vartheta$ -periodic of period  $k$  if and only if  $\psi(\alpha)$  is  $s$ -periodic of period  $k$ .*

*Proof.* Suppose firstly that  $\alpha$  is  $\vartheta$ -periodic of period  $k$ . Then,  $\psi \circ s^k \circ \psi(\alpha) = \alpha$ , implying that  $s^k(\psi(\alpha)) = \psi(\alpha)$ . If  $s^l(\psi(\alpha)) = \psi(\alpha)$ , for some integer  $l < k$ , then  $\psi \circ s^l \circ \psi(\alpha) = \alpha$ , namely  $\vartheta^l(\alpha) = \alpha$ . This is absurd, since  $k$  is the period of  $\alpha$  with respect to the map  $\vartheta$ .

Viceversa, suppose that  $\psi(\alpha)$  is  $s$ -periodic of period  $k$ . Then,  $s^k(\psi(\alpha)) = \psi(\alpha)$ , which implies that  $\psi \circ s^k \circ \psi(\alpha) = \alpha$ , namely  $\vartheta^k(\alpha) = \alpha$ . If  $\vartheta^l(\alpha) = \alpha$  for some positive integer  $l$  smaller than  $k$ , then  $s^l(\psi(\alpha)) = \psi(\alpha)$ , in contradiction with the fact that  $\psi(\alpha)$  is  $s$ -periodic of period  $k$ .  $\square$

We prove some properties of  $\vartheta$ -periodic elements.

**Lemma 2.2.** *The following hold.*

- The elements  $1, -1$  and  $\infty$  are  $\vartheta$ -periodic of period 1, while the element 0 is not  $\vartheta$ -periodic.
- If  $\alpha \in \mathbf{P}^1(\mathbf{F}_q) \setminus \{1, -1, \infty\}$  is  $\vartheta$ -periodic of period  $k$ , then  $\psi(\alpha)$  is  $s$ -periodic of period  $k$ , the multiplicative order  $d$  of  $\psi(\alpha)$  in  $\mathbf{F}_q^*$  is odd and  $k$  is equal to the multiplicative order  $\text{ord}_d(2)$  of 2 in  $(\mathbf{Z}/d\mathbf{Z})^*$ .
- If  $\beta \in \mathbf{F}_q \setminus \{0, 1\}$  has odd multiplicative order  $d$  in  $\mathbf{F}_q^*$ , then  $\beta$  is  $s$ -periodic of period  $k$ , where  $k$  is equal to  $\text{ord}_d(2)$ . Moreover,  $\alpha = \psi(\beta)$  is  $\vartheta$ -periodic of period  $k$ .

*Proof.* Firstly, consider the elements  $1, -1, \infty$ . Since  $\vartheta(-1) = -1$ ,  $\vartheta(1) = 1$  and  $\vartheta(\infty) = \infty$ , the statements about  $1, -1$  and  $\infty$  are proved. As regards the element 0, we note that  $\vartheta(0) = \infty$  and  $\vartheta(\infty) = \infty$ . It follows that 0 is not  $\vartheta$ -periodic.

By Lemma 2.1, if  $\alpha \in \mathbf{P}^1(\mathbf{F}_q) \setminus \{-1, 1, \infty\}$  is  $\vartheta$ -periodic of period  $k$ , then  $\beta = \psi(\alpha)$  is  $s$ -periodic of period  $k$  and  $s^k(\beta) = \beta^{2^k} = \beta$ , namely  $\beta^{2^k-1} = 1$ . Therefore,  $d$  divides  $2^k - 1$ , implying that  $d$  is odd. If  $r = \text{ord}_d(2)$ , then  $d \mid 2^r - 1$  and  $\beta^{2^r-1} = 1$  in  $\mathbf{F}_q^*$ , implying that  $s^r(\beta) = \beta$  and  $\vartheta^r(\alpha) = \alpha$ . Since  $d \mid 2^k - 1$ ,  $d \mid 2^r - 1$  and  $r = \text{ord}_d(2)$

we deduce that  $r \leq k$ . Moreover, by the fact that  $\vartheta^r(\alpha) = \alpha$  and  $\alpha$  is  $\vartheta$ -periodic of period  $k$ , we conclude that  $r \geq k$ . Therefore  $r = k$ .

Viceversa, take  $\beta \in \mathbf{F}_q \setminus \{0, 1\}$  such that the multiplicative order  $d$  of  $\beta$  in  $\mathbf{F}_q^*$  is odd. If  $k = \text{ord}_d(2)$ , then  $d \mid 2^k - 1$  and  $\beta^{2^k - 1} = 1$ . Therefore,  $\beta$  is  $s$ -periodic of period  $r \leq k$ . If  $s^r(\beta) = \beta$ , then  $\beta^{2^r - 1} = 1$ . Hence,  $d \mid 2^r - 1$ . Since  $k = \text{ord}_d(2)$ , we have that  $r = k$ . Therefore,  $\beta$  and  $\alpha = \psi(\beta)$  are respectively  $s$ - and  $\vartheta$ -periodic of period  $k$ .  $\square$

In the following Theorem the lengths and the number of cycles of  $Gr_q$  are given.

**Theorem 2.3.** *Let  $D = \{d_1, \dots, d_m\}$  be the set of the distinct odd integers greater than 1 which divide  $q - 1$ . For any  $1 \leq i \leq m$  denote by  $\text{ord}_{d_i}(2)$  the multiplicative order of 2 in  $(\mathbf{Z}/d_i\mathbf{Z})^*$ . Consider the set*

$$L = \{\text{ord}_{d_i}(2) : 1 \leq i \leq m\} = \{l_1, \dots, l_r\}$$

*of cardinality  $r$ , where  $0 \leq r \leq m$ , and the map*

$$\begin{aligned} l : D &\rightarrow L \\ d_i &\mapsto \text{ord}_{d_i}(2). \end{aligned}$$

*Then:*

- $L \cap \{1\} = \emptyset$ ;
- *the length of a cycle in  $Gr_q$  is a positive integer belonging to  $L \cup \{1\}$ ;*
- *there are exactly three cycles of length 1 in  $Gr_q$ ;*
- *for any integer  $k$  such that  $1 \leq k \leq r$  there are*

$$c_k = \frac{1}{l_k} \cdot \sum_{d_i \in l^{-1}(l_k)} \varphi(d_i)$$

*cycles of length  $l_k$  in  $Gr_q$ , being  $\varphi$  the Euler's totient function;*

- *the number of connected components of the graph  $Gr_q$  is*

$$3 + \sum_{k=1}^r c_k.$$

*Proof.* Since any element of  $L$  is equal to  $\text{ord}_d(2)$  for some odd integer  $d > 1$ , then 1 is not contained in  $L$ .

In Lemma 2.2 we proved that  $1, -1$  and  $\infty$  are  $\vartheta$ -periodic of period 1. Therefore these elements form three cycles of length 1. By Lemma 2.2 a  $\vartheta$ -periodic element of  $\mathbf{P}^1(\mathbf{F}_q) \setminus \{-1, 1, \infty\}$  has period  $k$ , where  $k$  is the multiplicative order of 2 in  $(\mathbf{Z}/d\mathbf{Z})^*$ , for some odd integer  $d$  which divides  $q - 1$ . Therefore, the length of a cycle is an integer belonging to  $L \cup \{1\}$ .

Take an odd divisor  $d_i > 1$  of  $q - 1$ . In  $\mathbf{F}_q^*$  there are  $\varphi(d_i)$  elements of multiplicative order  $d_i$ . Since  $\psi$  is a bijection on  $\mathbf{P}^1(\mathbf{F}_q)$ , each of these elements is of the form  $\psi(\alpha)$  for some  $\alpha \in \mathbf{P}^1(\mathbf{F}_q)$ . Let  $\beta = \psi(\alpha)$  be one of these elements. Since the multiplicative order of  $\beta$  in  $\mathbf{F}_q^*$  is greater

than 1, we have that  $\beta \neq 1$ . Therefore, by Lemma 2.2,  $\psi(\beta) = \alpha$  is  $\vartheta$ -periodic of period  $\text{ord}_{d_i}(2)$ . This means that, for any odd divisor  $d_i$  of  $q - 1$ , there are  $\varphi(d_i)$  elements in  $\mathbf{F}_q^*$  which are  $\vartheta$ -periodic of period  $\text{ord}_{d_i}(2)$ .

Consider now an element  $l_k \in L$ . Since  $\text{ord}_{d_i}(2) = l_k$  if and only if  $d_i \in l^{-1}(l_k)$ , the number of cycles of length  $l_k$  is given by  $c_k$ . Moreover, being any element of  $\mathbf{P}^1(\mathbf{F}_q)$  finally periodic, we conclude that the number of connected components of the graph is equal to the number of the cycles.  $\square$

We aim at describing the trees rooted in  $\vartheta$ -periodic elements of  $\mathbf{P}^1(\mathbf{F}_q)$ . Before proceeding, we note that the elements 1 and  $-1$  are not roots of any tree. In fact,

$$\begin{aligned} \vartheta(x) = 1 &\Leftrightarrow x + x^{-1} = 2 \Leftrightarrow (x - 1)^2 = 0 \Leftrightarrow x = 1 \\ \vartheta(x) = -1 &\Leftrightarrow x + x^{-1} = -2 \Leftrightarrow (x + 1)^2 = 0 \Leftrightarrow x = -1. \end{aligned}$$

We prove the following preliminary result.

**Lemma 2.4.** *Let  $\gamma \in \mathbf{F}_q$  be a non- $\vartheta$ -periodic point (in particular  $\gamma \notin \{1, -1\}$ ). Then,  $\vartheta(x) = \gamma$  for exactly two distinct elements  $x \in \mathbf{F}_q$ , provided that  $\text{ord}(\psi(\gamma)) \not\equiv 0 \pmod{2^e}$ , where  $\text{ord}(\psi(\gamma))$  is the multiplicative order of  $\psi(\gamma)$  in  $\mathbf{F}_q^*$ . If, on the contrary,  $\text{ord}(\psi(\gamma)) \equiv 0 \pmod{2^e}$ , then there is no  $x \in \mathbf{F}_q$  such that  $\vartheta(x) = \gamma$ .*

*Proof.* Take  $\gamma$  as in the hypotheses. We note that, if  $\vartheta(x) = \gamma$ , then  $x \notin \{-1, 0, 1\}$ , since  $\vartheta(-1) = -1$ ,  $\vartheta(1) = 1$  and  $\vartheta(0) = \infty$ , but  $\gamma \in \mathbf{F}_q \setminus \{1, -1\}$ . Hence, there exists  $x \in \mathbf{F}_q$  such that  $\vartheta(x) = \gamma$  iff  $\psi \circ s \circ \psi(x) = \gamma$ , namely iff  $\psi(x)^2 = \psi(\gamma)$ . This is equivalent to saying that  $\psi(\gamma)$  is a square in  $\mathbf{F}_q$ . This is true iff  $\psi(\gamma)^{(q-1)/2} = 1$  in  $\mathbf{F}_q^*$ , namely iff  $\text{ord}(\psi(\gamma)) \mid \frac{q-1}{2}$ . This latter is equivalent to saying that  $\text{ord}(\psi(\gamma)) \not\equiv 0 \pmod{2^e}$ .  $\square$

In the following result the depth of the reversed binary trees rooted in  $\vartheta$ -periodic elements is given.

**Theorem 2.5.** *Let  $\alpha \in \mathbf{P}^1(\mathbf{F}_q) \setminus \{1, -1\}$  be a  $\vartheta$ -periodic point. Then,  $\alpha$  is the root of a reversed binary tree of depth  $e$  in  $\text{Gr}_q$  with the following properties:*

- there are  $2^{k-1}$  vertices at any level  $1 \leq k \leq e$ ;
- the root  $\alpha$  has one child while the vertices at any level  $0 < k < e$  have two children;
- $\nu_2(\text{ord}(\psi(\beta))) = k$ , if  $\beta \in \mathbf{F}_q$  belongs to the level  $k > 0$  of the tree.

*Proof.* If  $\alpha = \infty$ , then  $\alpha$  is  $\vartheta$ -periodic of period 1. Moreover  $\vartheta(x) = \infty$  iff  $x = \infty$  or 0. The point 0 is the only vertex belonging to the first level of the tree rooted in  $\infty$ . Moreover,  $\psi(0) = -1$ , which has multiplicative order 2 in  $\mathbf{F}_q^*$ .

If  $\alpha \in \mathbf{F}_q \setminus \{-1, 1\}$ , then  $\psi(\alpha) \in \mathbf{F}_q^*$  and finding all the elements  $\beta$  in  $\mathbf{F}_q$  such that  $\vartheta(\beta) = \alpha$  amounts to finding all the elements  $\beta$  such that  $\psi \circ s \circ \psi(\beta) = \alpha$ . This latter is equivalent to  $s(\psi(\beta)) = \psi(\alpha)$ , namely  $\psi(\beta)^2 = \psi(\alpha)$ . According to Lemma 2.2 the order of  $\psi(\alpha)$  is odd. Hence,  $(\psi(\alpha))^{(q-1)/2} = 1$  in  $\mathbf{F}_q$ . Therefore,  $\psi(\alpha)$  is a square in  $\mathbf{F}_q^*$  and there are two distinct roots  $r_1, r_2 = -r_1$  of  $x^2 - \psi(\alpha)$  in  $\mathbf{F}_q$ . Being the map  $\psi$  a bijection on  $\mathbf{P}^1(\mathbf{F}_q)$ , it follows that  $r_1 = \psi(\beta_1)$  and  $r_2 = \psi(\beta_2)$  for two distinct elements  $\beta_1$  and  $\beta_2$  in  $\mathbf{F}_q$ . Moreover, since  $\alpha$  is  $\vartheta$ -periodic, one among  $\beta_1$  and  $\beta_2$ , say  $\beta_1$ , is  $\vartheta$ -periodic too and consequently  $r_1$  has odd order. On the contrary  $\beta_2$  is not  $\vartheta$ -periodic and  $\text{ord}(\psi(\beta_2)) = 2 \cdot \text{ord}(\psi(\beta_1))$ , namely  $\nu_2(\text{ord}(\psi(\beta_2))) = 1$ .

The remaining statements regarding the levels  $k > 0$  will be proved by induction on  $k$ . Consider firstly the level  $k = 1$ . If  $e = 1$ , then there is no element at the second level of the tree by Lemma 2.4. In the case  $e > 1$  consider the only element  $\gamma$  belonging to the first level of the tree rooted in  $\alpha$ . We have proved that  $\nu_2(\text{ord}(\psi(\gamma))) = 1$ . In virtue of Lemma 2.4 there are exactly two elements belonging to the level 2 of the tree, whose image under the action of the map  $\vartheta$  is  $\gamma$ .

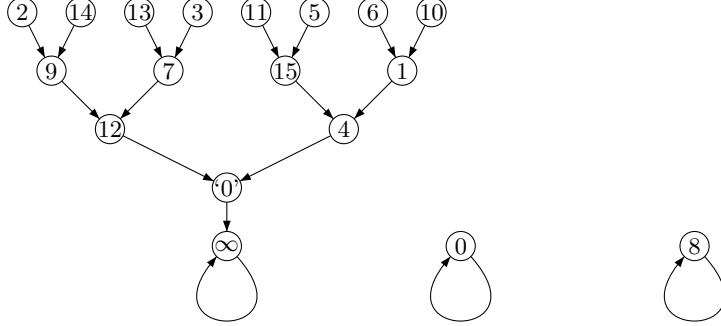
Now we proceed with the inductive step. Suppose that for some integer  $k > 1$  such that  $k-1 < e$  there are  $2^{k-2}$  elements at the level  $k-1$  of the tree and that each of these elements has two children. Moreover, if  $\gamma$  is one of the elements at the level  $k-1$ , then  $\nu_2(\text{ord}(\psi(\gamma))) = k-1$ . Let  $\beta$  be any of the children of  $\gamma$ . Since  $\vartheta(\beta) = \gamma$ , we have that  $\psi(\beta)^2 = \psi(\gamma)$ . Then  $\nu_2(\text{ord}(\psi(\beta))) = k$ . In addition, by Lemma 2.4  $\beta$  has no child if  $k = e$ , while it has 2 children if  $k < e$ . Finally, since  $2^{k-2}$  vertices belong to the level  $k-1$  of the tree and each of these vertices has two children, there are  $2^{k-1}$  vertices at the level  $k$  of the tree.  $\square$

In the following we construct and analyse the graphs  $Gr_{17}$ ,  $Gr_{23}$  and  $Gr_{49}$ .

**Example 2.6.** Let  $\alpha$  be the root of the Conway polynomial  $x - 3 \in \mathbf{F}_{17}[x]$ . Then,

$$\mathbf{P}^1(\mathbf{F}_{17}) = \{\alpha^i : 0 \leq i \leq 15\} \cup \{0, \infty\}.$$

We construct the graph  $Gr_{17}$  labelling the vertex  $\alpha^i$ , for  $0 \leq i \leq 15$ , with the corresponding exponent  $i$ . Moreover, we will use the symbol ‘0’ to denote the zero element of  $\mathbf{F}_{17}$ .

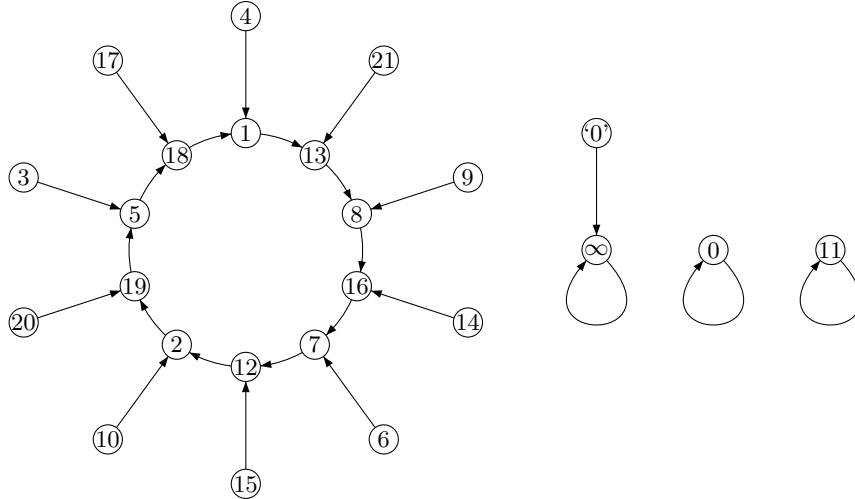


Using the notation of Theorem 2.3 we have that  $D = L = \emptyset$ . Therefore the graph  $Gr_{17}$  is formed just by cycles of length 1, namely the three cycles formed by  $1, -1$  and  $\infty$  (respectively  $0, 8$  and  $\infty$  with the notation introduced above). By Theorem 2.5, the vertex  $\infty$  is root of a tree of depth  $e = 4$ , being  $\nu_2(16) = 4$ .

**Example 2.7.** Let  $\alpha$  be the root of the Conway polynomial  $x - 5 \in \mathbf{F}_{23}[x]$ . Then,

$$\mathbf{P}^1(\mathbf{F}_{23}) = \{\alpha^i : 0 \leq i \leq 21\} \cup \{0, \infty\}.$$

We construct the graph  $Gr_{23}$  labelling the vertex  $\alpha^i$ , for  $0 \leq i \leq 21$ , with the corresponding exponent  $i$ . Moreover, we will use the symbol ' $0$ ' to denote the zero element of  $\mathbf{F}_{23}$ .

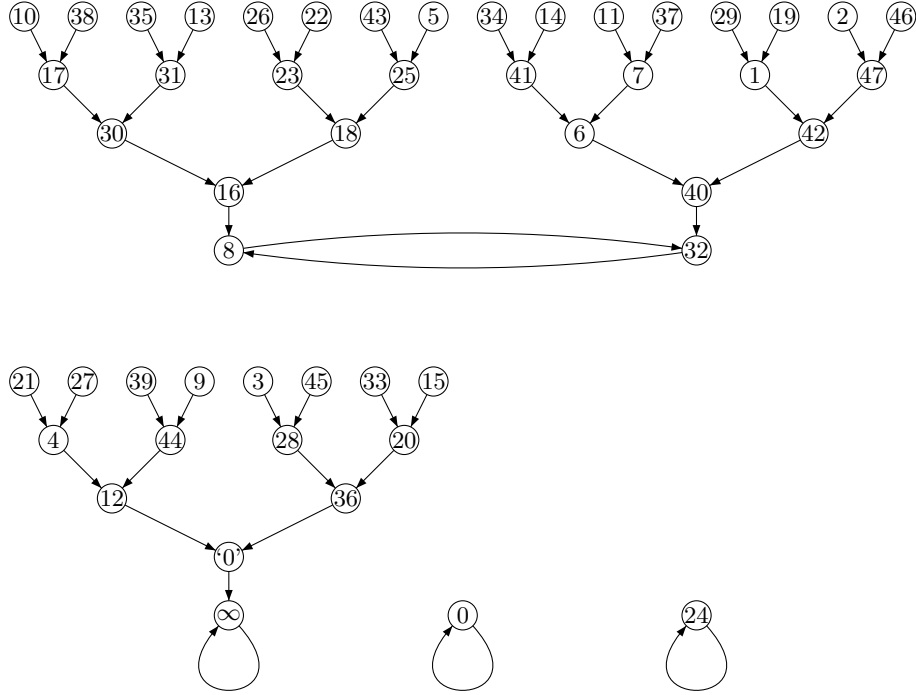


Using the notation of Theorem 2.3 we have that  $D = \{11\}$  and  $L = \{10\}$ . Therefore the graph  $Gr_{23}$  is formed just by three cycles of length 1 and by one cycle of length 10. By Theorem 2.5 any  $\vartheta$ -periodic point different from  $1$  and  $-1$  is root of a tree of depth  $e = 1$ , being  $e = \nu_2(22) = 1$ .

**Example 2.8.** Let  $\alpha$  be a root of the Conway polynomial  $x^2 - x + 3 \in \mathbf{F}_7[x]$ . Then,

$$\mathbf{P}^1(\mathbf{F}_{49}) = \{\alpha^i : 0 \leq i \leq 47\} \cup \{0, \infty\}.$$

We construct the graph  $Gr_{49}$  labelling the vertex  $\alpha^i$ , for  $0 \leq i \leq 47$ , with the corresponding exponent  $i$ . Moreover, we will use the symbol ‘0’ to denote the zero element of  $\mathbf{F}_7$ .



Using the notation of Theorem 2.3 we have that  $D = \{3\}$  and  $L = \{2\}$ . Therefore the graph  $Gr_{49}$  is formed by three cycles of length 1 and one cycle of length 2. By Theorem 2.5, any  $\vartheta$ -periodic point different from  $\pm 1$  is root of a reversed binary tree of depth  $e = 4$ , being  $\nu_2(48) = 4$ .

### 3. CONSTRUCTING SEQUENCES OF IRREDUCIBLE POLYNOMIALS

For the sake of clearness we introduce some notation.

**Definition 3.1.** If  $f \in \mathbf{F}_p[x]$ , for some odd prime  $p$ , is a monic irreducible polynomial with a root  $\alpha \neq 0$  in an appropriate extension of  $\mathbf{F}_p$ , then we define  $f^\vartheta$  as the minimal polynomial of  $\vartheta(\alpha)$ .

**Definition 3.2.** If  $p$  is an odd prime, then we denote by  $\text{Irr}_p$  the set of all monic irreducible polynomials of  $\mathbf{F}_p[x]$  different from  $x + 1$  and  $x - 1$ . If  $n$  is a positive integer, then  $\text{Irr}_p(n)$  denotes the set of all polynomials of  $\text{Irr}_p$  of degree  $n$ .



**Definition 3.3.** If  $f$  is a polynomial of positive degree  $n$  in  $\mathbf{F}_p[x]$ , then

$$f^{\tilde{Q}}(x) = 2^n \cdot x^n \cdot f(\vartheta(x)).$$

Consider the following Lemma.

**Lemma 3.4.** *Let  $p$  be an odd prime and  $n$  a positive integer. Suppose that  $\nu_2(p^n - 1) = k$  for some integer  $k \geq 2$ . Then,  $\nu_2(p^{2n} - 1) = k + 1$ .*

*Proof.* Since  $k \geq 2$  and  $p^n \equiv 1 \pmod{2^k}$ , it follows that  $p^n \equiv 1 \pmod{4}$  and  $p^n + 1 \equiv 2 \pmod{4}$ . Summing all up,

$$\begin{aligned} p^n - 1 &= 2^k \cdot m_1 \\ p^n + 1 &= 2 \cdot m_2 \end{aligned}$$

for some odd integers  $m_1, m_2$ .

Therefore,

$$p^{2n} - 1 = (p^n - 1) \cdot (p^n + 1) = 2^{k+1} \cdot m_1 \cdot m_2$$

and the thesis follows.  $\square$

*Remark 3.5.* We want to notice that the assumption about  $k$  in Lemma 3.4 cannot be dropped off. In fact, if  $k = 1$ , then anything can happen. Consider for example the primes 23 and 31 with  $n = 1$ . We have that  $\nu_2(22) = 1$  and  $\nu_2(30) = 1$ . Nevertheless,

$$\begin{aligned} \nu_2(23^2 - 1) &= \nu_2(528) = 4 \\ \nu_2(31^2 - 1) &= \nu_2(960) = 6. \end{aligned}$$

We will make use of the following technical Lemma in the forthcoming Theorem.

**Lemma 3.6.** *Let  $f$  be a polynomial of positive degree  $n$  of  $\mathbf{F}_p[x]$ , for some odd prime  $p$ . Suppose that  $\beta$  is a root of  $f$  and that  $\beta = \vartheta(\alpha)$  for some  $\alpha, \beta$  in suitable extensions of  $\mathbf{F}_p$ . Then,  $\alpha$  and  $\alpha^{-1}$  are roots of  $f^{\tilde{Q}}$ .*

*Proof.* The thesis follows easily evaluating  $f^{\tilde{Q}}$  at  $\alpha$  and  $\alpha^{-1}$ . In fact,

$$\begin{aligned} f^{\tilde{Q}}(\alpha) &= 2^n \cdot \alpha^n \cdot f(\vartheta(\alpha)) \\ f^{\tilde{Q}}(\alpha^{-1}) &= 2^n \cdot \alpha^{-n} \cdot f(\vartheta(\alpha)) \end{aligned}$$

and  $f(\vartheta(\alpha)) = f(\beta) = 0$ .  $\square$

**Theorem 3.7.** *Let  $f$  be a polynomial of  $\text{Irr}_p(n)$  for some odd prime  $p$  and positive integer  $n$ . Suppose that  $f(x) \neq x$ , if  $n = 1$ . The following hold.*

- *If the set of roots of  $f$  is not closed under inversion, then  $f^{\tilde{\vartheta}} \in \text{Irr}_p(n)$ .*
- *If the set of roots of  $f$  is closed under inversion, then  $n$  is even and  $f^{\tilde{\vartheta}} \in \text{Irr}_p(n/2)$ .*

*Proof.* Suppose that the set of roots of  $f$  is not closed under inversion. If  $\beta = \vartheta(\alpha)$ , for some root  $\alpha$  of  $f$ , then  $\beta$  is root of  $f^{\tilde{\vartheta}}$ . Since  $\alpha$  is root of the polynomial  $x^2 - 2\beta x + 1$  and the degree of  $\alpha$  over  $\mathbf{F}_p$  is  $n$ , the degree of  $\beta$  over  $\mathbf{F}_p$  is either  $n$  or  $n/2$ . In the former case  $f^{\tilde{\vartheta}} \in \text{Irr}_p(n)$ . In the latter case,  $f^{\tilde{\vartheta}}$  has degree  $n/2$  (and  $n$  is even). Consider the polynomial  $g = (f^{\tilde{\vartheta}})^{\tilde{Q}}$ . The polynomial  $g$  has degree  $n$  and  $\alpha, \alpha^{-1}$  are among its roots. We deduce that  $g$  is the minimal polynomial of  $\alpha$ , namely  $g = f$ . This implies that the set of roots of  $f$  is inverse-closed in contradiction with the initial assumption. Therefore,  $f^{\tilde{\vartheta}} \in \text{Irr}_p(n)$ .

Suppose now that the set of roots of  $f$  is inverse-closed and consider any root  $\alpha$  of  $f$ . Since  $\alpha = \alpha^{-1}$  if and only if  $\alpha^2 = 1$ , namely  $\alpha = \pm 1$ , and  $f$  is different from  $x + 1$  and  $x - 1$ , we conclude that  $\alpha \neq \alpha^{-1}$  and the degree of  $f$  is even. We know that the set of roots of  $f$  is equal to  $\{\alpha^{p^i} : 0 \leq i \leq n-1\}$ . Let  $\beta = \vartheta(\alpha)$ . By definition  $f^{\tilde{\vartheta}}$  is the minimal polynomial of  $\beta$ . Hence any root of  $f^{\tilde{\vartheta}}$  is of the form  $\beta^{p^i}$  for some integer  $i$ . We notice that

$$\beta^{p^i} = \vartheta(\alpha)^{p^i} = \frac{1}{2^{p^i}} \cdot (\alpha + \alpha^{-1})^{p^i} = \frac{1}{2} \cdot (\alpha^{p^i} + \alpha^{-p^i}) = \vartheta(\alpha^{p^i}).$$

Therefore, the map  $\vartheta$  defines a surjective 2-1 correspondence between the set of roots of  $f$  and the set of roots of  $f^{\tilde{\vartheta}}$ , implying that the degree of  $f^{\tilde{\vartheta}}$  is  $n/2$ .  $\square$

**Theorem 3.8.** *Suppose that  $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  belongs to  $\text{Irr}_p(n)$  for some positive integer  $n$  and odd prime  $p$ . The following hold.*

- 0 is not root of  $f^{\tilde{Q}}$ .
- The set of roots of  $f^{\tilde{Q}}$  is closed under inversion.
- Either  $f^{\tilde{Q}} \in \text{Irr}_p(2n)$ , or  $f^{\tilde{Q}}$  splits into the product of two polynomials  $m_\alpha, m_{\alpha^{-1}}$  in  $\text{Irr}_p(n)$ , which are respectively the minimal polynomial of  $\alpha$  and  $\alpha^{-1}$ , for some  $\alpha \in \mathbf{F}_{p^n}$ . Moreover, in the latter case, at least one among  $\alpha$  and  $\alpha^{-1}$  is not  $\vartheta$ -periodic.

*Proof.* Since  $f$  is irreducible, the coefficient  $a_0 = 0$  only if  $f(x) = x$ . If this is the case, then  $f^{\tilde{Q}}(x) = x^2 + 1$  and this latter polynomial does not vanish for  $x = 0$ . If  $a_0 \neq 0$ , then the constant term of  $f^{\tilde{Q}}$  is equal to 1. In fact, since

$$f^{\tilde{Q}}(x) = 2^n \cdot x^n \cdot (2^{-n}(x + x^{-1})^n + a_{n-1} \cdot 2^{-n+1}(x + x^{-1})^{n-1} + \dots + a_0),$$

we have that the constant term of  $f^{\tilde{Q}}$  is determined by the expansion of  $(x + x^{-1})^n$  only. Therefore, the constant term of  $f^{\tilde{Q}}$  is equal to 1 and we deduce that 0 cannot be a root of  $f^{\tilde{Q}}$ .

Take now a root  $\alpha$  of  $f^{\tilde{Q}}$ . Since  $\alpha \neq 0$ , there exists the inverse  $\alpha^{-1}$ . Being  $f^{\tilde{Q}}(\alpha) = 0$  and  $\alpha \neq 0$ , we get that  $f(\vartheta(\alpha)) = 0$ . Since

$f^{\tilde{Q}}(\alpha^{-1}) = 2^n \cdot \alpha^{-n} \cdot f(\vartheta(\alpha))$ , we get that also  $f^{\tilde{Q}}(\alpha^{-1}) = 0$ . We conclude that the set of roots of  $f^{\tilde{Q}}$  is closed under inversion.

We know that  $f$  is irreducible of degree  $n$ . Hence, if  $\beta \in \mathbf{F}_{p^n}$  is any root of  $f$ , then the set of roots of  $f$  is  $\{\beta^{p^i} : 0 \leq i \leq n-1\}$ . Consider now a root  $\alpha$  of  $f^{\tilde{Q}}$ . Since  $\alpha \neq 0$ , we have that  $f(\vartheta(\alpha)) = 0$ , namely  $\vartheta(\alpha)$  is a root of  $f$ . Let  $\gamma = \vartheta(\alpha)$ , where  $\gamma = \beta^{p^i}$ , for some  $0 \leq i \leq n-1$ . We notice that  $\vartheta(\alpha) = \gamma$  is equivalent to saying that  $\alpha$  is root of the polynomial  $x^2 - 2\gamma x + 1$ . Summing all up, we conclude that the degree of  $\alpha$  over  $\mathbf{F}_p$  is either  $n$  or  $2n$ . In the latter case the minimal polynomial of  $\alpha$  has degree  $2n$  and must be equal to  $f^{\tilde{Q}}$ , which has degree  $2n$ , is monic and has  $\alpha$  among its roots. In the former case the minimal polynomial  $m_\alpha$  of  $\alpha$  has degree  $n$  and by Theorem 3.7 the set of roots of  $m_\alpha$  is not inverse-closed (on the contrary,  $f$ , which is the minimal polynomial of  $\vartheta(\alpha)$ , should have degree  $n/2$ ). Hence, using the notation of the claim,  $f^{\tilde{Q}}(x) = m_\alpha(x) \cdot m_{\alpha^{-1}}(x)$  for some  $\alpha \in \mathbf{F}_{p^n}$ . Moreover, we observe that  $\vartheta(x) = \gamma$  if and only if  $x = \alpha$  or  $\alpha^{-1}$ . If  $\gamma$  is not  $\vartheta$ -periodic, then both  $\alpha$  and  $\alpha^{-1}$  cannot be  $\vartheta$ -periodic too. On the converse, one among  $\alpha$  and  $\alpha^{-1}$  is  $\vartheta$ -periodic, while the other element belongs to the level 1 of the tree rooted in  $\gamma$ .  $\square$

**3.1. A procedure for constructing sequences of irreducible polynomials.** The following Theorem furnishes a procedure for constructing an infinite sequence of irreducible polynomials, starting from any polynomial of  $\text{Irr}_p(n)$ .

**Theorem 3.9.** *Let  $f_0 \in \text{Irr}_p(n)$ , where  $p$  is an odd prime,  $n$  a positive integer,  $\nu_2(p^n - 1) = e_0$  and  $\nu_2(p^{2n} - 1) = e_1$  for some positive integers  $e_0, e_1$  with  $e_0 < e_1$ .*

*If  $f_0^{\tilde{Q}}$  is irreducible, define  $f_1 := f_0^{\tilde{Q}}$ . Otherwise, set  $f_1$  equal to one of the monic irreducible factors of degree  $n$  having a root which is not  $\vartheta$ -periodic, as stated in Theorem 3.8.*

*For  $i \geq 2$  define inductively a sequence of polynomials  $\{f_i\}_{i \geq 2}$  in such a way: if  $f_{i-1}^{\tilde{Q}}$  is irreducible, then  $f_i := f_{i-1}^{\tilde{Q}}$ ; otherwise, set  $f_i$  equal to one of the two irreducible factors of degree  $n$  of  $f_{i-1}^{\tilde{Q}}$  as stated in Theorem 3.8.*

*Then, there exist two positive integers  $s_1, s_2$  such that:*

- $f_0, \dots, f_{s_1-1} \in \text{Irr}_p(n)$ ;
- $f_{s_1}, \dots, f_{s_1+s_2-1} \in \text{Irr}_p(2n)$ ;
- $f_{s_1+s_2+i} \in \text{Irr}_p(2^{2+i}n)$  for any  $i \geq 0$ ;
- $s_1 \leq e_0 + 1$  and  $s_2 = e_1 - e_0$ .

*Proof.* Let  $\beta_0 \in \mathbf{F}_{p^n}$  be a root of  $f_0$ . In  $Gr_{p^n}$  the vertex  $\beta_0$  lies on the level  $k \geq 0$  of some binary tree of depth  $e_0$  rooted in an element

$\gamma \in \mathbf{F}_{p^n}$ . In particular, if  $k = 0$ , then  $\beta_0 = \gamma$ . We distinguish two cases based upon the reducibility of  $f_0^{\tilde{Q}}$ .

- If  $f_0^{\tilde{Q}}$  is irreducible of degree  $2n$ , then we set  $f_1 := f_0^{\tilde{Q}}$ . The equation  $\vartheta(x) = \beta_0$  has exactly two solutions  $\beta_1$  and  $\beta_1^{-1}$  in  $\mathbf{F}_{p^{2n}}$ . By Lemma 3.6  $\beta_1$  and  $\beta_1^{-1}$  are roots of  $f_1 \in \text{Irr}_p(2n)$ . All considered we can say that  $\beta_0$  is a leaf of  $Gr_{p^n}$ . Therefore, in this case  $k = e_0$  and  $s_1 = 1$ .
- If  $f_0^{\tilde{Q}}$  is not irreducible, then we define  $f_1$  as one of the monic irreducible factors of degree  $n$  of  $f_0^{\tilde{Q}}$  having a root  $\beta_1$ , which is not  $\vartheta$ -periodic. We prove that, for any integer  $i$  such that  $0 \leq i \leq e_0 - k$ , there exists an element  $\beta_i \in \mathbf{F}_{p^n}$  such that  $\vartheta^{k+i}(\beta_i) = \gamma$  and  $\beta_i$  is a root of  $f_i$ . Indeed, this is trivially true if  $i = 0$ . Suppose that, for some  $i < e_0 - k$ , there exists an element  $\beta_i$  such that  $\vartheta^{k+i}(\beta_i) = \gamma$  and  $\beta_i \in \mathbf{F}_{p^n}$  is a root of  $f_i$ . Since  $k + i < e_0$  and the tree rooted in  $\gamma$  has depth  $e_0$ , there exists an element  $\beta' \in \mathbf{F}_{p^n}$  such that  $\vartheta(\beta') = \beta_i$ . By Lemma 3.6 the element  $\beta'$  is a root of  $f_i^{\tilde{Q}}$ . Since  $\beta' \in \mathbf{F}_{p^n}$ , the polynomial  $f_i^{\tilde{Q}}$  splits into the product of two polynomials  $g_1, g_2 \in \text{Irr}_p(n)$ . One among  $g_1$  and  $g_2$  is equal to  $f_{i+1}$ . Moreover, either  $\beta'$  or  $(\beta')^{-1}$  is root of  $f_{i+1}$ . We can say, without loss of generality, that  $\beta'$  is root of  $f_{i+1}$ . Therefore, setting  $\beta_{i+1} := \beta'$ , we get that  $\beta_{i+1}$  is a root of  $f_{i+1}$  and  $\vartheta^{k+i+1}(\beta_{i+1}) = \gamma$ . Now, consider the polynomial  $f_{e_0-k}$ . By construction  $\beta_{e_0-k} \in \mathbf{F}_{p^n}$  is a root of  $f_{e_0-k}$ . Moreover,  $\beta_{e_0-k}$  is a leaf of the tree of  $Gr_{p^n}$  rooted in  $\gamma$ . Consider now an element  $\beta$  such that  $\vartheta(\beta) = \beta_{e_0-k}$ . Since  $\beta$  cannot belong to the same tree of  $Gr_{p^n}$ , we have that  $\beta \in \mathbf{F}_{p^{2n}} \setminus \mathbf{F}_{p^n}$ . Therefore,  $f_{s_1} := f_{e_0-k+1} = f_{e_0-k}^{\tilde{Q}}$  is irreducible of degree  $2n$ . Since  $k \geq 0$ , the index  $s_1 = e_0 - k + 1 \leq e_0 + 1$ .

Now we prove, by induction on  $i$ , that, for any integer  $i$  such that  $e_0 - k + 1 \leq i \leq e_1 - k$ , there exists an element  $\beta_i \in \mathbf{F}_{p^{2n}}$  such that  $\vartheta^{k+i}(\beta_i) = \gamma$  and  $\beta_i$  is a root of  $f_i$ . In virtue of what we have just proved, this is true if  $i = e_0 - k + 1$ . Take now an integer  $e_0 - k + 1 \leq i < e_1 - k$ . By inductive hypothesis there exists an element  $\beta_i \in \mathbf{F}_{p^{2n}}$  such that  $\vartheta^{k+i}(\beta_i) = \gamma$  and  $\beta_i$  is a root of  $f_i$ . Since  $\beta_i$  belongs to the level  $i+k < e_1$  of the tree of  $Gr_{p^{2n}}$  rooted in  $\gamma$ , there exists an element  $\beta' \in \mathbf{F}_{p^{2n}}$  such that  $\vartheta(\beta') = \beta_i$ . Then, one among  $\beta'$  and  $(\beta')^{-1}$ , say  $\beta'$ , is root of  $f_{i+1}$ . We set  $\beta_{i+1} := \beta'$  and complete the inductive proof.

Finally, since  $\beta_{e_1-k}$  is a leaf of the tree of  $Gr_{p^{2n}}$  rooted in  $\gamma$ , any element  $\beta$  such that  $\vartheta(\beta) = \beta_{e_1-k}$  cannot belong to  $\mathbf{F}_{p^{2n}}$ . Therefore such a  $\beta$  must belong to  $\mathbf{F}_{p^{4n}}$  and its minimal polynomial  $f_{s_1+s_2} := f_{e_1-k+1}$  has degree  $4n$ . In particular,  $s_1 + s_2 = e_1 - k + 1$ . We remind that  $s_1 = e_0 - k + 1$  and conclude that  $s_2 = e_1 - e_0$ .  $\square$

*Remark 3.10.* Using the notation of Theorem 3.9, if  $f_0^{\tilde{Q}}$  is not irreducible, then it splits into the product of two irreducible polynomials  $g_1, g_2$  of equal degree. By Theorem 3.8 one of them will have a root which is not  $\vartheta$ -periodic. In principle we do not know which of the two polynomials has this property. Therefore, we just set  $f_1$  equal to either  $g_1$  or  $g_2$ . Then we proceed constructing the sequence as stated in the Theorem's claim. If none of the polynomials  $f_i$ , for  $i \leq e_0 + 1$ , has degree  $2n$ , then we break the procedure and set  $f_1 := g_2$ , which will have a non- $\vartheta$ -periodic root, as stated by Theorem 3.8 (see Example 3.12).

We conclude this Section with two examples of sequences with initial polynomial  $f_0$  belonging respectively to  $\text{Irr}_7(1)$  and  $\text{Irr}_7(2)$ .

**Example 3.11.** Let  $f_0 := x \in \text{Irr}_7(1)$ . The only root of  $f_0$  is the zero element '0' of  $\mathbf{F}_7$ . Since  $\vartheta('0') = \vartheta^2('0') = \infty$ , we have that '0' is not  $\vartheta$ -periodic. Therefore, if  $f_0^{\tilde{Q}}$  is not irreducible, then it splits into the product of two irreducible factors of degree 1 which have no  $\vartheta$ -periodic roots.

The polynomial  $f_1(x) := f_0^{\tilde{Q}}(x) = x^2 + 1$  is irreducible. Indeed, this is an accordance with the fact that  $\nu_2(6) = 1$  and the root of  $f_0$  belongs to the level 1 of the tree of  $Gr_7$  rooted in  $\infty$ , namely '0' is a leaf of the tree.

Now we notice that  $\nu_2(48) = 4$ . With the notation of Theorem 3.9 we have that  $e_0 = 1$ , while  $e_1 = 4$ . Therefore we expect that  $f_2$  and  $f_3$  belong to  $\text{Irr}_7(2)$ , while  $f_4$  belongs to  $\text{Irr}_7(4)$ . The polynomial  $f_1^{\tilde{Q}}(x) = x^4 - x^2 + 1$  splits into the product of two irreducible factors, namely  $f_1^{\tilde{Q}}(x) = (x^2 + 2) \cdot (x^2 + 4)$ . We set  $f_2(x) := x^2 + 2$ . Now,  $f_2^{\tilde{Q}}(x) = x^4 + 3x^2 + 1$  splits into the product of two irreducible factors as  $f_2^{\tilde{Q}}(x) = (x^2 + 3x - 1) \cdot (x^2 + 4x - 1)$ . We set  $f_3(x) := x^2 + 3x - 1$ .

The polynomial  $f_3^{\tilde{Q}}(x) = x^4 - x^3 - 2x^2 - x + 1$  is irreducible. Hence we set  $f_4 := f_3^{\tilde{Q}}$ . Now, for  $i \geq 3$ , any polynomial  $f_{i+1} := f_i^{\tilde{Q}}$  is irreducible, namely we can construct an infinite sequence of irreducible polynomials whose degree doubles at each step.

**Example 3.12.** Let  $f_0 := x - 3 \in \text{Irr}_7(1)$ . Since  $\nu_2(6) = 1$ , using the notation of Theorem 3.9 we have that  $s_1 \leq 2$ . Therefore, in the sequence we are going to construct, at most the polynomials  $f_0$  and  $f_1$  have degree 1. The polynomial  $f_0^{\tilde{Q}}(x) = x^2 + x + 1$  is not irreducible. Indeed,  $f_0^{\tilde{Q}}(x) = (x - 4)(x - 2)$ . We set  $f_1$  equal to one among the two factors of degree 1 of  $f_0^{\tilde{Q}}$ . For example, set  $f_1 := x - 4$ . The polynomial  $f_1^{\tilde{Q}} = x^2 - x + 1$  factors as  $f_1^{\tilde{Q}} = (x - 3)(x - 5)$ . If we set  $f_2$  equal to any of the factors of degree 1 of  $f_1^{\tilde{Q}}$  we get that  $f_2$  is a polynomial of degree

1 too. Hence, we break the procedure and change the polynomial  $f_1$  as suggested in Remark 3.10.

Set  $f_1 := x - 2$ . Now,  $f_1^{\tilde{Q}} = x^2 + 3x + 1$  is irreducible in  $\mathbf{F}_7[x]$ . Therefore we set  $f_2(x) := x^2 + 3x + 1$ . Now,  $f_2^{\tilde{Q}}(x) = x^4 - x^3 - x^2 - x + 1$ . We notice that  $f_2^{\tilde{Q}} = (x^2 + x + 3) \cdot (x^2 - 2x - 2)$ , where both the factors of degree two belong to  $\text{Irr}_7(2)$ . Set  $f_3(x) := x^2 + x + 3$ . Now,  $f_3^{\tilde{Q}}(x) = x^4 + 2x^3 + 2x + 1$  splits into the product of two irreducible polynomials of  $\text{Irr}_7(2)$ , namely  $f_3^{\tilde{Q}}(x) = (x^2 - 3x - 2) \cdot (x^2 - 2x + 3)$ . Set  $f_4(x) := x^2 - 3x - 2$ . We have that  $f_4^{\tilde{Q}}(x) = x^4 + x^3 + x^2 + 1 \in \text{Irr}_7(4)$ . Therefore we set  $f_5 := f_4^{\tilde{Q}}$  and, in virtue of Theorem 3.9, we are guaranteed that any polynomial  $f_{i+1} := f_i^{\tilde{Q}}$  will be irreducible for  $i \geq 4$ .

#### 4. A NOTE ABOUT THEOREM 3.9

In a generic step of the iterative procedure described in Theorem 3.9 we have to decide if the polynomial  $f_i^{\tilde{Q}}$  is irreducible or not and, in the latter case, factoring it.

Dropping the indices off, the problem we are dealing with consists in deciding if, taken an irreducible polynomial  $f$  of degree  $n$  of  $\mathbf{F}_p[x]$  where  $p$  is an odd prime and  $n$  a positive integer, the polynomial  $f^{\tilde{Q}}$  is irreducible or not. In the latter case we have to find two irreducible monic polynomials, say  $g_1$  and  $g_2$ , of degree  $n$  such that  $f^{\tilde{Q}}(x) = g_1(x) \cdot g_2(x)$ .

If  $\beta \in \mathbf{F}_{p^n}$  is a root of  $f$ , then any element of  $\mathbf{F}_{p^n}$  is expressible as

$$(4.1) \quad c_{n-1}\beta^{n-1} + \cdots + c_1\beta + c_0,$$

where  $c_{n-1}, \dots, c_1, c_0 \in \mathbf{F}_p$ .

If  $\alpha$  is a solution of the equation  $\vartheta(x) = \beta$ , then  $\alpha$  is a root of  $f^{\tilde{Q}}$ . In fact,

$$f^{\tilde{Q}}(\alpha) = 2^n \cdot \alpha^n \cdot f(\vartheta(\alpha)) = 0,$$

since  $\vartheta(\alpha) = \beta$  is a root of  $f$ . The fact that  $\vartheta(\alpha) = \beta$  is equivalent to saying that  $\alpha$  is root of

$$x^2 - 2\beta \cdot x + 1,$$

namely

$$(4.2) \quad \alpha = \beta + \sqrt{\beta^2 - 1} \quad \text{or} \quad \alpha = \beta - \sqrt{\beta^2 - 1}$$

for some square root  $\sqrt{\beta^2 - 1}$  of  $\beta^2 - 1$ . Therefore, either  $\alpha \in \mathbf{F}_{p^n}$  or  $\alpha \in \mathbf{F}_{p^{2n}}$ . In particular,  $\alpha \in \mathbf{F}_{p^n}$  if and only if  $\beta^2 - 1$  is a square in  $\mathbf{F}_{p^n}$ , namely if and only if  $(\beta^2 - 1)^{\frac{p^n - 1}{2}} = 1$  in  $\mathbf{F}_{p^n} \cong \mathbf{F}_p[x]/(f)$ . If this latter test fails, then we can conclude that  $f^{\tilde{Q}}$  is irreducible. On the contrary, we can find a square root of  $\beta^2 - 1$  as explained for example in the proof of Lemma 7.7 of [vdW06], which relies upon Theorem VI.6.1

of [Lan02]. To do that, set  $a = \beta^2 - 1$ . Following the steps of the proof we define  $A = a^{(p-1)/2}$  and look for a non-zero element  $c \in \mathbf{F}_{p^n}$  such that

$$(4.3) \quad c^p = Ac.$$

Since any  $c \in \mathbf{F}_{p^n}$  can be expressed as in (4.1), solving the last equation amounts to finding the coefficients  $c_i \in \mathbf{F}_p$  which satisfy the equation

$$(4.4) \quad c_{n-1}\beta^{p(n-1)} + \cdots + c_1\beta^p + c_0 = A \cdot (c_{n-1}\beta^{n-1} + \cdots + c_1\beta + c_0).$$

Any exponent in the powers of  $\beta$  can be reduced to a positive integer smaller than  $n$ , since  $f(\beta) = 0$  and  $f$  has degree  $n$ . Therefore, solving (4.4) amounts to finding a solution

$$(c_0, \dots, c_{n-1}) \in \mathbf{F}_p^n$$

of a linear system of at most  $n$  linear equations. Once we have found such a  $c$ , we notice that  $c^2/a$  is a quadratic residue in  $\mathbf{F}_p$ . Finally, we find a square root  $d$  of  $c^2/a$  in  $\mathbf{F}_p$  and notice that  $c/d$  is a square root of  $a$ .

Summing all up,  $c/d \in \mathbf{F}_{p^n}$  can be expressed as a linear combination of  $1, \beta, \beta^2, \dots, \beta^{n-1}$  with coefficients in  $\mathbf{F}_p$ . Substituting  $c/d$  in place of  $\sqrt{\beta^2 - 1}$  we express  $\alpha$  as linear combination of the powers  $\beta^i$  with  $0 \leq i \leq n-1$ .

To end with, we can factor  $f^{\tilde{Q}}$  as the product of two irreducible factors  $g_1(x), g_2(x) \in \mathbf{F}_p[x]$  of degree  $n$ , namely

$$g_1(x) = \prod_{i=0}^{n-1} (x - \alpha^{p^i}) \quad g_2(x) = \prod_{i=0}^{n-1} (x - (\alpha^{-1})^{p^i}).$$

**Example 4.1.** Let  $f(x) = x^3 + 3x^2 + 2 \in \mathbf{F}_5[x]$ . Then,

$$f^{\tilde{Q}}(x) = x^6 + x^5 + 3x^4 + 3x^3 + 3x^2 + x + 1.$$

Following the steps explained above we want to decide if  $f^{\tilde{Q}}$  is irreducible or not and, in the latter case, factor it. Let  $\beta$  be a root of  $f$ . We know that  $f^{\tilde{Q}}$  is irreducible if and only if  $\beta^2 - 1$  is not a square in  $\mathbf{F}_{5^3} \cong \mathbf{F}_5[x]/(f)$ . Since

$$(x^2 - 1)^{62} = 1 \quad \text{in } \mathbf{F}_5[x]/(f),$$

we conclude that  $a = \beta^2 - 1$  is a quadratic residue in  $\mathbf{F}_{5^3}$ . Therefore,  $f^{\tilde{Q}}(x) = g_1(x) \cdot g_2(x)$ , where  $g_1, g_2$  are two monic irreducible polynomials of degree 3 of  $\mathbf{F}_5[x]$ . Aiming to find the polynomials  $g_1, g_2$ , we look for an element

$$c = c_2\beta^2 + c_1\beta + c_0 \in \mathbf{F}_{125}^*$$

such that  $c^5 = A \cdot c$ , where  $A = a^2$  and  $a = \beta^2 - 1$ . Expanding the last equation our problem is equivalent to finding three coefficients  $c_0, c_1, c_2 \in \mathbf{F}_5$ , not simultaneously equal to zero, such that

$$c_2 \cdot \beta^{10} + c_1\beta^5 + c_0 = A \cdot (c_2\beta^2 + c_1\beta + c_0).$$

Expanding the left hand side of the last equation we get

$$c_2 \cdot (3\beta^2 + 2\beta + 1) + c_1 \cdot (\beta^2 + \beta + 2) + c_0,$$

while expanding the right hand side we get

$$c_2 \cdot (\beta^2 + \beta + 1) + c_1 \cdot (2\beta^2 + 2\beta + 1) + c_0 \cdot (2\beta^2 + 3\beta + 2).$$

Therefore, solving  $c^5 = A \cdot c$  amounts to solving the following linear system over  $\mathbf{F}_5$ :

$$\begin{cases} 2c_2 - c_1 - 2c_0 &= 0 \\ c_2 - c_1 - 3c_0 &= 0 \\ c_1 - c_0 &= 0 \end{cases}$$

It is easily seen that such a system has infinitely many solutions. More precisely, for a free choice of  $c_0 \in \mathbf{F}_5$ , the other coefficients are uniquely determined as  $c_1 = c_0$  and  $c_2 = -c_0$ . For example, choosing  $c_0 = 1$  we get  $c_1 = 1$  and  $c_2 = -1$ . Hence,

$$c = -\beta^2 + \beta + 1 \quad \text{and} \quad \frac{c^2}{a} = 4.$$

A square root of 4 in  $\mathbf{F}_5$  is  $d = 2$ . Therefore,

$$\frac{c}{d} = 2\beta^2 - 2\beta - 2$$

is a square root of  $\beta^2 - 1$ . We conclude that

$$\alpha = 2\beta^2 - \beta - 2$$

is a solution of  $\vartheta(x) = \beta$ . Therefore, the polynomial

$$g_1(x) = (x - \alpha) \cdot (x - \alpha^5) \cdot (x - \alpha^{25}) = x^3 + 3x + 3$$

is a monic irreducible factor of  $f^{\tilde{Q}}$ . Now we can easily find the other factor and conclude that

$$f^{\tilde{Q}}(x) = g_1(x) \cdot g_2(x) = (x^3 + 3x + 3) \cdot (x^3 + x^2 + 2).$$

## REFERENCES

- [Lan02] S. Lang, *Algebra*, Springer-Verlag, 2002.
- [Ugo11a] S. Ugolini, *Graphs associated with the map  $x \mapsto x + x^{-1}$  in finite fields of characteristic five*, arxiv (2011), <http://arxiv.org/abs/1110.0968>.
- [Ugo11b] ———, *Graphs associated with the map  $x \mapsto x + x^{-1}$  in finite fields of characteristic three*, arxiv (2011), <http://arxiv.org/abs/1108.1763>.
- [Ugo12] ———, *Graphs associated with the map  $x \mapsto x + x^{-1}$  in finite fields of characteristic two*, Proceedings of the 10<sup>th</sup> International Conference on Finite Fields and their Applications, Contemporary Mathematics, 579, AMS (2012).
- [vdW06] C. van de Woestijne, *Deterministic equation solving over finite fields*, Doctoral thesis, Leiden University, 2006, <http://hdl.handle.net/1887/4392>.

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